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Fourier transform in multimode systems in the Bargmann representation

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Abstract

A Fourier transform in a multimode system is studied, using the Bargmann representation. The growth of a Bargmann function is shown to be related to the second-order correlation of the corresponding state. Both the total growth and the total second-order correlation remain unchanged under the Fourier transform. Examples with coherent states, squeezed states and Mittag–Leffler states are discussed.

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1. Introduction

The study of various aspects of unitary transformations in multimode systems is very important. In this paper we consider a d-mode system with creation and annihilation operators $(a_0^\dagger, a_0), \ldots, (a_{d-1}^\dagger, a_{d-1})$. We study a unitary transformation which transforms them into $(b_0^\dagger, b_0), \ldots, (b_{d-1}^\dagger, b_{d-1})$, where the $\{b_M\}$ are related to $\{a_M\}$ through a Fourier transform and the $\{b_M^\dagger\}$ are also related to $\{a_M^\dagger\}$ through a Fourier transform. This Fourier transform is a non-local transformation that involves all modes and is very different from a local Fourier transform on a particular mode, which relates the position and momentum states in this mode.

There exists extensive literature on applied aspects of devices that perform this Fourier transform [1–6] and in this paper we study theoretical aspects using the theory of analytic functions of many complex variables. In section 2 we present the basic formalism of d harmonic oscillators and define the notation. In section 3 we define Bargmann functions [7] for multimode systems. In section 4 we discuss the growth of Bargmann functions of one complex variable and in section 5 the growth of Bargmann functions of many complex variables. We explain that the mathematical concept of growth is related to the physical quantity of second-order correlation.

In section 6 we discuss the Fourier transform using the language of Bargmann functions. We explain that these transformations do not change the total growth of Bargmann functions and that they leave invariant the total second-order correlation of a state. The usual properties of Fourier transforms are expressed in the present context in section 7.

These general ideas are discussed explicitly for the examples of coherent states, squeezed states and Mittag–Leffler states. In section 8 we consider coherent states and show that the transformed states are also coherent states. In section 9 we consider squeezed states. For factorizable squeezed states, the transformed states are entangled.

In section 10 we consider an example which involves Mittag–Leffler functions. This is important for theoretical reasons because the corresponding Bargmann function can have any given growth. We conclude in section 11 with a discussion of our results.

2. *d* harmonic oscillators

We consider a system comprised of d harmonic oscillators with Hilbert space $H = \mathcal{H}\otimes\cdots\otimes\mathcal{H}$. We also consider the creation and annihilation operators

$$\begin{aligned} a_M^\dagger &= \mathbf{1}\otimes\cdots\otimes a^\dagger\otimes\cdots\otimes\mathbf{1} \\ a_M &= \mathbf{1}\otimes\cdots\otimes a\otimes\cdots\otimes\mathbf{1} \\ \left[a_M, a_N^\dagger\right] &= \delta_{MN}\mathbf{1}, \end{aligned} \tag{1}$$

where M belongs in Z_d (the integers modulo d). The displacement operators are given by

$$D_M(z) = \exp\left(za_M^\dagger - z^* a_M\right), \tag{2}$$

where z is a complex number. Coherent states are defined as

$$\begin{aligned} |z_0, \ldots, z_{d-1}\rangle &= [D_0(z_0)\ldots D_{d-1}(z_{d-1})]\,|0,\ldots,0\rangle \\ &= \exp\left(-\frac{1}{2}\sum_{M=0}^{\infty}|z_M|^2\right)\sum\frac{z_0^{N_0}\ldots z_{d-1}^{N_{d-1}}}{(N_0!\ldots N_{d-1}!)^{1/2}}|N_0,\ldots,N_{d-1}\rangle, \end{aligned} \tag{3}$$

where $|0,\ldots,0\rangle$ is the d-mode vacuum state. In the sum (and also in the sums in equations (4)–(7) below) the integer variables $N_0, \ldots, N_{d-1}$ take values from 0 to ∞.

We consider an arbitrary state

$$|f\rangle = \sum F(N_0,\ldots,N_{d-1})|N_0,\ldots,N_{d-1}\rangle; \qquad \sum|F(N_0,\ldots,N_{d-1})|^2 = 1. \tag{4}$$

We use the notation

$$\begin{aligned} |f^*\rangle &= \sum[F(N_0,\ldots,N_{d-1})]^*|N_0,\ldots,N_{d-1}\rangle, \\ \langle f^*| &= \sum F(N_0,\ldots,N_{d-1})\langle N_0,\ldots,N_{d-1}|. \end{aligned} \tag{5}$$

For later use we define the average number of photons and the second-order correlation for the mode M:

$$\begin{aligned} \langle N_M\rangle &= \sum N_M|F(N_0,\ldots,N_{d-1})|^2, \\ g_M^{(2)} &= \frac{\sum N_M^2|F(N_0,\ldots,N_{d-1})|^2 - \langle N_M\rangle}{\langle N_M\rangle^2}. \end{aligned} \tag{6}$$

We also define the total average number of photons and the total second-order correlation

$$\begin{aligned} \langle N_T\rangle &= \sum N_T|F(N_0,\ldots,N_{d-1})|^2; \qquad N_T = N_0+\cdots+N_{d-1}, \\ g_T^{(2)} &= \frac{\sum N_T^2|F_0(N_0,\ldots,N_{d-1})|^2 - \langle N_T\rangle}{\langle N_T\rangle^2}. \end{aligned} \tag{7}$$

3. Bargmann representation for *d* harmonic oscillators

The Bargmann representation for the state of equation (4) is given by

$$f(\{z_M\}) \equiv f(z_0, \ldots, z_{d-1}) = \exp\left(\frac{1}{2}\sum_{M=0}^{\infty}|z_M|^2\right)\langle f^*|z_0, \ldots, z_{d-1}\rangle$$
$$= \sum F(N_0, \ldots, N_{d-1})\frac{z_0^{N_0}\ldots z_{d-1}^{N_{d-1}}}{(N_0!\ldots N_{d-1}!)^{1/2}}, \quad (8)$$

where in the sum the variables $N_0, \ldots, N_{d-1}$ take values from 0 to ∞. The function $f(z_0, \ldots, z_{d-1})$ is an analytic function of $z_0, \ldots, z_{d-1}$.

Equation (8) shows that if $(w_0, \ldots, w_{d-1})$ is a zero of the Bargmann function $f(\{z_M\})$, i.e., if

$$f(w_0, \ldots, w_{d-1}) = 0, \quad (9)$$

then the coherent state $|w_0, \ldots, w_{d-1}\rangle$ is orthogonal to the state $|f^*\rangle$.

The scalar product of two states $|f\rangle$ and $|g\rangle$ is given by

$$\langle f|g\rangle = \int [f(\{z_M\})]^* g(\{z_M\})\, \mathrm{d}\mu(\{z_M\}),$$
$$\mathrm{d}\mu(\{z_M\}) = \exp\left(-\sum_{M=0}^{\infty}|z_M|^2\right)\frac{d^2z_0}{\pi}\ldots\frac{d^2z_{d-1}}{\pi}. \quad (10)$$

In the Bargmann representation the creation and annihilation operators are given by

$$a_M^\dagger \to z_M, \qquad a_M \to \partial_{z_M}. \quad (11)$$

4. Growth of Bargmann functions of one variable

The growth of analytic functions of one complex variable and its relation to the density of zeros of this function have been discussed extensively in the mathematical literature [8]. These ideas have been used in the context of Bargmann functions in quantum mechanics in [9].

Let $f(z)$ be an entire function of one complex variable z and let $M(R)$ be the maximum value of $|f(z)|$ on the circle $|z| = R$. The growth of $f(z)$ is characterized by the order ρ and the type σ which are defined as

$$\rho = \lim_{R\to\infty}\sup\frac{\ln\ln M(R)}{\ln R}, \qquad \sigma = \lim_{R\to\infty}\sup\frac{\ln M(R)}{R^\rho}. \quad (12)$$

In a simple language, this means that as $|z| \to \infty$ the function grows like

$$|f(z)| \approx \exp(\sigma|z|^\rho). \quad (13)$$

For Bargmann functions of one variable, convergence of the scalar product leads to the result [9] that either $\rho < 2$ (in which case σ can take any value) or $\rho = 2$ and $\sigma < 1/2$.

The growth of an entire function is related to the rate of decrease of its Taylor coefficients. If

$$f(z) = \sum_{N=0}^{\infty} c_N z^N \quad (14)$$

then the order of the growth is given by

$$\rho = \lim_{N\to\infty}\frac{N\ln N}{-\ln|c_N|}. \quad (15)$$

We have seen in equation (8) that the Bargmann function of the state

$$|f\rangle = \sum_{N=0}^{\infty} F(N)|N\rangle, \qquad \sum_{N=0}^{\infty} |F(N)|^2 = 1 \tag{16}$$

is

$$f(z) = \sum_{N=0}^{\infty} \frac{F(N)z^N}{(N!)^{1/2}}. \tag{17}$$

Therefore $c_N = F(N)(N!)^{-1/2}$ and taking into account Stirling's formula for the asymptotic behaviour of $N!$, we conclude that at large N

$$|F(N)|^2 \approx N^{-\lambda N}, \qquad \lambda = \frac{2}{\rho} - 1. \tag{18}$$

4.1. Growth and bunching

Here we make a connection between the mathematical concept of growth and the second-order correlation $g^{(2)}$ which is a quantity of physical interest.

The second-order correlation for the state of equation (16) is

$$g^{(2)} = \frac{\sum N^2|F(N)|^2 - \langle N\rangle}{\langle N\rangle^2}, \qquad \langle N\rangle = \sum_{N=0}^{\infty} N|F(N)|^2. \tag{19}$$

When $g^{(2)} < 1$ the state shows antibunching (i.e., regular arrival of photons in a detector). In the opposite limit $g^{(2)} \gg 1$ the state shows strong bunching (i.e., irregular arrival of photons in a detector).

Comparison of equations (18), (19) shows that there is a link between the growth of a Bargmann function and the corresponding second-order correlation $g^{(2)}$. We stress that it is a 'weak link' because equation (18) refers to very large N, while equation (19) is, in practice, based on smaller N. Nevertheless, given two states with the same average number of photons $\langle N\rangle$, we expect the one with the larger order to have larger $g^{(2)}$. Therefore an 'approximate statement' is that Bargmann functions with small (close to zero) or large (close to 2) order show small bunching (small $g^{(2)}$) or strong bunching (large $g^{(2)}$), correspondingly. Numerical results presented later will support this statement.

There is a well-known connection between the growth of an entire function and the density of its zeros. This has been used in [9] to derive results about the completeness of sequences of coherent states.

5. Growth of Bargmann functions of many variables

There is an enormous amount of work on the growth of analytic functions of one complex variable [8], but there is less work on the growth of analytic functions of several complex variables (for a summary, see [10]). There are various definitions for various purposes and in the present context we are interested in the convergence of the integral in the scalar product of equation (10). For this reason, we consider the 'total growth' (i.e., with respect to all variables) and we use the norm $\left(\sum |z_M|^2\right)^{1/2}$. Let $M(R)$ be the maximum value of $|f(\{z_M\})|$ on the sphere

$$\left(\sum_{M=0}^{\infty} |z_M|^2\right)^{1/2} = R. \tag{20}$$

The order ρ_T and the type σ_T of the total growth of $f(\{z_M\})$ are defined as in equation (12). This means that as $\sum |z_M|^2 \to \infty$ the function grows like

$$|f(\{z_M\})| \approx \exp\left[\sigma_T \left(\sum_{M=0}^{\infty} |z_M|^2\right)^{\rho_T/2}\right]. \tag{21}$$

It is easily seen from equation (10) that in order to have convergence of the integral in the scalar product, the total growth of Bargmann functions should be either $\rho_T < 2$ (in which case σ_T can take any value) or $\rho_T = 2$ and $\sigma_T < 1/2$.

5.1. Factorizable functions

We consider the factorizable states

$$|f\rangle = |f_0\rangle \otimes \cdots \otimes |f_{d-1}\rangle = \sum F_0(N_0)\ldots F_{d-1}(N_{d-1})|N_0,\ldots,N_{d-1}\rangle, \tag{22}$$

where, in the summation, the variables $N_0,\ldots,N_{d-1}$ take values from 0 to ∞. The corresponding Bargmann functions $f(\{z_M\})$ are factorizable:

$$f(\{z_M\}) = f_0(z_0)\ldots f_{d-1}(z_{d-1}). \tag{23}$$

We want to relate the growths of the various factors $f_M(z_M)$ with the total growth of the function $f(\{z_M\})$. Let (ρ_M, σ_M) be the order and type of the growth of $f_M(z_M)$ (for all M). As $R \to \infty$ the variables $|z_M| = \lambda_M R \to \infty$, where λ_M are coefficients that depend on the direction which we follow as we go towards infinity. Then the function $f(\{z_M\})$ grows like

$$|f(\{z_M\})| \approx \exp\left(\sum_{M=0}^{\infty} \sigma_M |z_M|^{\rho_M}\right) \approx \exp\left(\sum_{M=0}^{\infty} \tau_M R^{\rho_M}\right), \qquad \tau_M = \sigma_M \lambda_M^{\rho_M}. \tag{24}$$

This shows that the order of the total growth of $f(\{z_M\})$ is the maximum of ρ_M:

$$\rho_T = \max(\rho_0,\ldots,\rho_{d-1}). \tag{25}$$

If only one of the ρ_M (e.g., the ρ_i) is equal to the maximum value, then the corresponding type σ_i is equal to the type of the total growth of $f(\{z_M\})$:

$$\sigma_T = \sigma_i. \tag{26}$$

If, however, two or more of the ρ_M (e.g., the $\rho_i = \rho_j = \cdots$) are equal to the maximum value, then the sum of the corresponding types is *greater than or equal* to the type of the total growth of $f(\{z_M\})$:

$$\sigma_T \leqslant \sigma_i + \sigma_j + \cdots. \tag{27}$$

We discuss this in the example with coherent states later.

We next make a connection between the total growth of factorizable functions, and the total second-order correlation, which is here given by

$$g_T^{(2)} = \frac{\sum (N_0 + \cdots + N_{d-1})^2 |F_0(N_0)\ldots F_{d-1}(N_{d-1})|^2 - \langle N_T\rangle}{\langle N_T\rangle^2}$$
$$\langle N_T\rangle = \left(\sum N_0 |F(N_0)|^2\right) + \cdots + \left(\sum N_{d-1}|F(N_{d-1})|^2\right), \tag{28}$$

where, in the summation, the variables $N_0,\ldots,N_{d-1}$ take values from 0 to ∞. Using equation (18) we find that for large N_M

$$|F_0(N_0)\ldots F_{d-1}(N_{d-1})|^2 \approx \prod_{M\in Z_d} N_M^{-\lambda_M N_M} \leqslant \left(\prod_{M\in Z_d} N_M^{N_M}\right)^{-\lambda_T}$$
$$\lambda_M = \frac{2}{\rho_M} - 1, \qquad \lambda_T = \frac{2}{\rho_T} - 1 = \min(\lambda_0,\ldots,\lambda_{d-1}). \tag{29}$$

We then use the inequality

$$\prod_{M\in Z_d} N_M^{N_M} \geqslant \left(\frac{N_T}{d}\right)^{N_T}, \qquad N_T = N_0 + \cdots + N_{d-1}, \tag{30}$$

to show that for large N_M

$$|F_0(N_0)\ldots F_{d-1}(N_{d-1})|^2 \approx \left(\frac{N_T}{d}\right)^{-\lambda_T N_T}. \tag{31}$$

This is similar to equation (18) for the one variable case. We can now make a connection between the order ρ_T of the total growth of a factorizable Bargmann function and the corresponding second-order correlation $g_T^{(2)}$. Bargmann functions with small or large order show small bunching or strong bunching, correspondingly.

6. Unitary transformations

We consider the unitary operator [1]

$$U = \exp\left[\mathrm{i}\sum_{M,K} a_M^\dagger \Phi_{MK} a_K\right], \qquad M, K \in Z_d, \tag{32}$$

where Φ is a $d \times d$ Hermitian matrix. It is known [11] that

$$\begin{aligned} b_M &\equiv U a_M U^\dagger = \sum_{K\in Z_d} G_{MK} a_K, \\ b_M^\dagger &\equiv U a_M^\dagger U^\dagger = \sum_{K\in Z_d} G_{MK}^* a_K^\dagger, \qquad [b_M, b_N^\dagger] = \delta_{MN}\mathbf{1}, \end{aligned} \tag{33}$$

where

$$G = \exp(-\mathrm{i}\Phi), \qquad GG^\dagger = \mathbf{1}. \tag{34}$$

The unitary transformations U leave invariant the vacuum state

$$U|0,\ldots,0\rangle = |0,\ldots,0\rangle. \tag{35}$$

They also preserve the total number of photons in a state:

$$\sum_{M\in Z_d} a_M^\dagger a_M = \sum_{M\in Z_d} b_M^\dagger b_M. \tag{36}$$

We note that symplectic $Sp(2d, R)$ transformations [12] are given by

$$\begin{aligned} b_M &\equiv V a_M V^\dagger = \sum_{K\in Z_d} \left(G_{MK} a_K + J_{MK} a_K^\dagger\right) \\ b_M^\dagger &\equiv V a_M^\dagger V^\dagger = \sum_{K\in Z_d} \left(G_{MK}^* a_K^\dagger + J_{MK}^* a_K\right) \\ [b_M, b_N^\dagger] &= \delta_{MN}\mathbf{1}, \end{aligned} \tag{37}$$

where

$$\sum_{K\in Z_d} (G_{MK} G_{NK}^* - J_{MK} J_{NK}^*) = \delta_{MN} \qquad \sum_{K\in Z_d} (G_{MK} J_{NK} - J_{MK} G_{NK}) = 0.$$

The transformations of equation (33) form a subgroup of $Sp(2d, R)$ (with $J_{MK} = 0$).

In the Bargmann representation equation (33) is written as

$$\partial_{\mathfrak{z}_M} = \sum_{K \in Z_d} G_{MK} \partial_{z_K}, \qquad \mathfrak{z}_M = \sum_{K \in Z_d} G^*_{MK} z_K. \tag{39}$$

We next show that for a coherent state

$$U|z_0, \ldots, z_{d-1}\rangle = |\zeta_0, \ldots, \zeta_{d-1}\rangle, \qquad \zeta_M = \sum_{K \in Z_d} z_K G^*_{KM}. \tag{40}$$

In general G_{KM} is not equal to G_{MK} and for this reason we use different notation for the variables $\mathfrak{z}_M$ and ζ_M. In the special case of symmetric matrix G, these two variables are the same.

It is easily seen that

$$\sum_{M \in Z_d} |\zeta_M|^2 = \sum_{K \in Z_d} |z_K|^2. \tag{41}$$

Using this and equation (40) we show that if $f(\{z_M\})$ is the Bargmann function of a state $|f\rangle$ then the $f(\{\zeta_M\}) \equiv \varphi(\{z_M\})$ is the Bargmann function of the state $U|f\rangle$. We express this as

$$Uf(\{z_M\}) = f(\{\zeta_M\}) \equiv \varphi(\{z_M\}), \qquad \zeta_M = \sum_{K \in Z_d} z_K G^*_{KM}. \tag{42}$$

The Jacobian of the transformation is equal to 1:

$$\frac{\partial(\{z_M, z^*_M\})}{\partial(\{\zeta_M, \zeta^*_M\})} = |\det G|^2 = 1. \tag{43}$$

This, together with equation (41), shows that

$$\mathrm{d}\mu(\{z_M\}) = \mathrm{d}\mu(\{\zeta_M\}). \tag{44}$$

We note that the transformations U do not change the total second-order correlation $g_T^{(2)}$ defined in equation (7). This is easily seen using equation (36).

With regard to the growth, equation (41) implies that the function $M(R)$ is the same for both Bargmann functions $f(\{z_M\})$ and $\varphi(\{z_M\})$. Therefore the unitary transformations U of equation (33) do not change the total growth of the Bargmann function $f(\{z_M\})$. We stress that U is a special case of unitary transformations, where annihilation operators are transformed into a combination of annihilation operators, and creation operators are transformed into a combination of creation operators. General unitary transformations (such as the more general transformations V of equation (37)) do change the total growth of Bargmann functions.

6.1. Fourier transform

An important special case of these transformations is for

$$G = \mathcal{F}, \qquad \Phi = \mathrm{i} \ln \mathcal{F}, \tag{45}$$

where $\mathcal{F}$ is the $d \times d$ Fourier matrix

$$\mathcal{F}_{MK} = \frac{1}{\sqrt{d}} \omega^{MK}, \qquad \omega = \exp\left(\mathrm{i}\frac{2\pi}{d}\right). \tag{46}$$

In this case, the unitary operator of equation (32) is given by

$$U_F = \exp\left[-\sum_{M,K} a^\dagger_M (\ln \mathcal{F})_{MK} a_K\right], \qquad U_F^4 = \mathbf{1}, \qquad M, K \in Z_d, \tag{47}$$

and equation (33) reduces to

$$U_F a_M U_F^\dagger = d^{-1/2} \sum_{K \in Z_d} \omega^{MK} a_K, \qquad U_F a_M^\dagger U_F^\dagger = d^{-1/2} \sum_{K \in Z_d} \omega^{-MK} a_K^\dagger. \tag{48}$$

In the Bargmann representation this is expressed as

$$\partial_{\zeta_M} = d^{-1/2} \sum_{K \in Z_d} \omega^{MK} \partial_{z_K}, \qquad \zeta_M = d^{-1/2} \sum_{K \in Z_d} \omega^{-MK} z_K. \tag{49}$$

If $f(\{z_M\})$ is the Bargmann function of a state $|f\rangle$ then the $f(\{\zeta_M\}) \equiv \varphi(\{z_M\})$ is the Bargmann function of the state $U_F|f\rangle$:

$$U_F f(\{z_M\}) = f(\{\zeta_M\}) \equiv \varphi(\{z_M\}), \qquad \zeta_M = d^{-1/2} \sum_{K \in Z_d} \omega^{-MK} z_K. \tag{50}$$

We note that in this paper we use the logarithm of the Fourier matrix $\mathcal{F}$ of equation (46) to construct the operator U_F of equation (47) which leads to the transformations of equation (48), among the d oscillators. Recently [13] has studied fractional Fourier transforms in the context of finite systems. It interesting to use the logarithms of these matrices to generalize the operator U_F and the transformations of equation (48). We do not pursue this direction in the present paper.

7. Properties of the Fourier transform

The operator U_F has the 'standard' Fourier transform properties and below we discuss the most important ones. The property $U_F^4 = \mathbf{1}$ gives

$$U_F^4 f(\{z_M\}) = f(\{z_M\}). \tag{51}$$

The U_F^2 is a parity operator in the sense that

$$U_F^2 f(\{z_M\}) = f(\{z_{-M}\}). \tag{52}$$

Here the indices are integers modulo d, i.e., $f(\{z_{-M}\}) = f(\{z_{d-M}\})$.

We next use equation (50) to show that

$$U_F f(\{z_N - z_{N+M}\}) = f(\{(1 - \omega^{KM})\zeta_K\}). \tag{53}$$

This might be viewed as the analogue of the property that gives the Fourier transform of the derivative (discrete in the present context) of a function.

The convolution property in the present context is given by

$$U_F f(\{z_M u_M\}) = f(\{(\zeta * v)_M\}) \tag{54}$$

where

$$\begin{aligned} \zeta_M &= \frac{1}{\sqrt{d}} \sum_{K \in Z_d} \omega^{-MK} z_K; \qquad v_M = \frac{1}{\sqrt{d}} \sum_{K \in Z_d} \omega^{-MK} u_K \\ (\zeta * v)_M &\equiv \sum_{K \in Z_d} \zeta_{M-K} v_K. \end{aligned} \tag{55}$$

8. Coherent states

As an example we consider the coherent state $|w_0, \ldots, w_{d-1}\rangle$ which is represented with the Bargmann function

$$f(\{z_M\}) = \exp\left[\sum_{M\in Z_d} w_M z_M - \frac{1}{2}\sum_{M\in Z_d} |w_M|^2\right]. \tag{56}$$

This function is factorizable

$$f(\{z_M\}) = \prod_{M\in Z_d} f_M(z_M) \qquad f_M(z_M) = \exp\left[w_M z_M - \frac{1}{2}|w_M|^2\right]. \tag{57}$$

The factor $f_M(z_M)$ has growth with order $\rho = 1$ and $\sigma = |w_M|$. According to our general result earlier, the total growth has order and type

$$\rho_T = 1, \qquad \sigma_T \leqslant \sum_{M\in Z_d} |w_M|. \tag{58}$$

We confirm this with an explicit calculation. We consider the maximum value of $|f(\{z_M\})|$ on the sphere $\left(\sum |z_M|^2\right)^{1/2} = R$. We first note that

$$|f(\{z_M\})| \leqslant \exp\left[\sum_{M\in Z_d} |w_M z_M| + \frac{1}{2}\sum_{M\in Z_d} |w_M|^2\right]. \tag{59}$$

We use the simpler notation $r_M = |z_M|$ and we need to find the maximum of

$$g(r_0, \ldots, r_{d-2}) = |w_0|r_0 + \cdots + |w_{d-2}|r_{d-2} + |w_{d-1}|\left[R^2 - r_0^2 - \cdots - r_{d-2}^2\right]^{1/2}. \tag{60}$$

At the maximum

$$\frac{\partial g(\{r_M\})}{\partial r_M} = 0, \tag{61}$$

and we get a system of $d-1$ equations with $d-1$ unknowns which leads to the result

$$r_M = \frac{|w_M| R}{\left[\sum |w_M|^2\right]^{1/2}}. \tag{62}$$

We can check that the second derivatives are negative. Therefore the maximum value of $g(r_0, \ldots, r_{d-2})$ is $R\left[\sum |w_M|^2\right]^{1/2}$ and the total growth of the Bargmann function for coherent states (given in equation (56)) is

$$\rho_T = 1, \qquad \sigma_T = \left[\sum_{M\in Z_d} |w_M|^2\right]^{1/2}. \tag{63}$$

These results are consistent with those in equation (58) which are based on general arguments.

8.1. Fourier transform of coherent states

We consider the Fourier transform of the coherent state $|w_0, \ldots, w_{d-1}\rangle$. Its Bargmann function has been given in equation (56) and its Fourier transform is given by

$$U_F \exp\left[\sum_{M\in Z_d} w_M z_M - \frac{1}{2}\sum_{M\in Z_d}|w_M|^2\right] = \exp\left[\sum_{M\in Z_d} w_M \zeta_M - \frac{1}{2}\sum_{M\in Z_d}|w_M|^2\right]$$
$$= \exp\left[\sum_{M\in Z_d} u_M z_M - \frac{1}{2}\sum_{M\in Z_d}|u_M|^2\right] \qquad (64)$$

$$u_M = d^{-1/2}\sum_{K\in Z_d} w_K \omega^{KM}.$$

We note that both the original state of equation (56) and the transformed state of equation (64) are factorizable states.

We have seen in equation (63) that the total growth of the Bargmann function of equation (56) describing coherent states is $\rho_T = 1$ and $\sigma_T = \left[\sum |w_M|^2\right]^{1/2}$. As we explained earlier, the transformation U_F does not change the growth of the Bargmann function of a state. We confirm this here, because the growth of the transformed state of equation (64) is $\rho_T' = 1$ and $\sigma_T' = \left[\sum |u_M|^2\right]^{1/2}$ and it is easily seen that $\rho_T' = \rho_T$ and $\sigma_T' = \sigma_T$.

9. Squeezed states

9.1. One-mode squeezed states

One-mode squeezed states are defined as

$$|w_0; r, \theta\rangle = S(r,\theta)|w_0\rangle, \qquad (65)$$

where $S(r,\theta)$ is the squeezing operator:

$$S(r,\theta) = \exp\left[-\tfrac{1}{4} r\, \mathrm{e}^{-\mathrm{i}\theta}(a^\dagger)^2 + \tfrac{1}{4} r\, \mathrm{e}^{\mathrm{i}\theta} a^2\right]. \qquad (66)$$

The Bargmann function of this state is [9]

$$f(z_0) = \mathcal{L}\exp\left[-\frac{\alpha}{2} z_0^2 + \beta z_0\right] \qquad (67)$$

where

$$\alpha = \tanh\left(\frac{r}{2}\right)\mathrm{e}^{-\mathrm{i}\theta}, \qquad \beta = w_0(1-|\alpha|^2)^{1/2}$$
$$\mathcal{L} = (1-|\alpha|^2)^{1/4}\exp\left[\frac{1}{2}\alpha^* w_0^2 - \frac{1}{2}|w_0|^2\right]. \qquad (68)$$

It is easily seen that the growth of $f(z_0)$ has order $\rho = 2$ and type $\sigma = |\alpha|/2$.

We consider the simple case of $\arg(w_0) = 0$ and $\theta = 0$. In this case the average number of photons and the second-order correlation are given by

$$\langle N\rangle = w_0^2\left[\cosh\left(\frac{r}{2}\right) - \sinh\left(\frac{r}{2}\right)\right]^2 + \left[\sinh\left(\frac{r}{2}\right)\right]^2$$
$$g^{(2)} = 1 + \frac{e^{-r}-1}{\langle N\rangle} + \frac{1+\sinh r}{\langle N\rangle^2}\left[\sinh\left(\frac{r}{2}\right)\right]^2. \qquad (69)$$

In figure 1 we plot $g^{(2)}$ against r for $\langle N\rangle = 10$.

Squeezed states have been studied extensively in the literature because they have small values (less than 1) of the second-order correlation $g^{(2)}$ (antibunching). We stress, however, that this is the case only for a small window of the squeezing parameters r, θ. As seen in figure 1, there is a large window of these parameters ($r > 2.2$) where the $g^{(2)}$ takes very large values. And this is consistent with our 'approximate statement' made earlier, that states of high order, such as the squeezed states, will have large second-order correlations.

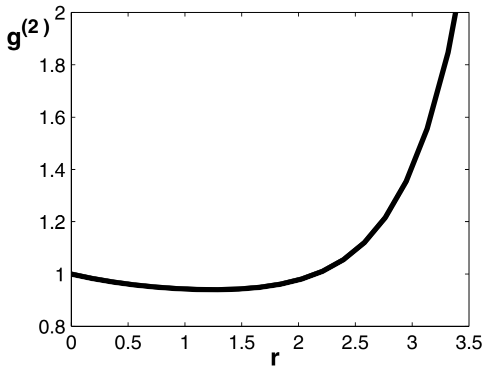

Figure 1. $g^{(2)}$ against r for the squeezed state of equation (65), with $\langle N\rangle = 10, \theta = 0$ and $\arg(w_0) = 0$.

9.2. *Multi-mode squeezed states*

For a d-mode squeezed state the Bargmann function is

$$f(\{z_K\}) = \mathcal{N}\exp\left[-\frac{1}{2}\sum_{M,N\in Z_d} A_{MN}z_M z_N + \sum_{M\in Z_d}\beta_M z_M\right]$$
$$A_{MN} = \alpha_M\delta_{MN}, \qquad \mathcal{N} = \mathcal{L}_0\ldots\mathcal{L}_{d-1}. \tag{70}$$

This is a factorizable function and, according to our general results earlier, the total growth of this function has order $\rho_T = 2$ and type $\sigma_T \leqslant \left(\sum|\alpha|_M\right)/2$.

Its Fourier transform is given by

$$U_F f(\{z_M\}) = \mathcal{N}\exp\left[-\frac{1}{2}\sum_{M,N\in Z_d} A_{MN}\zeta_M\zeta_N + \sum_{M\in Z_d}\beta_M\zeta_M\right]$$
$$= \mathcal{N}\exp\left[-\frac{1}{2}\sum_{M,N\in Z_d}\tilde{A}_{MN}z_M z_N + \sum_{M\in Z_d}\tilde{\beta}_M z_M\right], \tag{71}$$

where

$$\tilde{A}_{MN} = d^{-1}\sum_{K\in Z_d}\alpha_K\omega[-K(M+N)], \qquad \tilde{\beta}_M = d^{-1/2}\sum_{K\in Z_d}\beta_K\omega(-KM). \tag{72}$$

We note that in the original state of equation (70) the matrix A_{MN} is diagonal and therefore this state is factorizable. In its Fourier transform of equation (71), the matrix $\tilde{A}_{MN}$ is not diagonal and therefore this state is entangled.

The transformation U_F does not change the growth, and therefore the total growth of $U_F f(\{z_M\})$ has order $\rho_T = 2$ and type $\sigma_T \leqslant \left(\sum|\alpha|_M\right)/2$.

10. Mittag–Leffler states

The order ρ can take all values between 0 and 2. We show this explicitly by giving an example of a state which has Bargmann function with a given order ρ and given type σ.

The case of one mode has been discussed in [9]. We consider the state

$$|\rho, \sigma\rangle = \sum_{N=0}^{\infty} F(N)|N\rangle, \qquad F(N) = \mathcal{K}\frac{\sigma^{N/\rho}(N!)^{1/2}}{\Gamma\left(\frac{N}{\rho}+1\right)}, \tag{73}$$

where $\mathcal{K}$ is a normalization constant:

$$\mathcal{K} = \left[\sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{\left[\Gamma\left(\frac{N}{\rho}+1\right)\right]^2}\right]^{-\frac{1}{2}}. \tag{74}$$

The corresponding Bargmann function is given in terms of the Mittag–Leffler function [14, p 206]

$$f(z) = \mathcal{K} E_{1/\rho}(\sigma^{1/\rho} z), \tag{75}$$

and it has growth with order ρ and type σ.

We consider the d-mode state

$$|s\rangle = |\rho_0, \sigma_0\rangle \otimes \cdots \otimes |\rho_{d-1}, \sigma_{d-1}\rangle, \tag{76}$$

which is represented with the Bargmann function

$$f(\{z_M\}) = \left[\mathcal{K}_0 E_{1/\rho_0}\left(\sigma_0^{1/\rho_0} z_0\right)\right] \ldots \left[\mathcal{K}_{d-1} E_{1/\rho_{d-1}}\left(\sigma_{d-1}^{1/\rho_{d-1}} z_{d-1}\right)\right]. \tag{77}$$

This is a factorizable function and according to our general result in equation (25) the total growth has order $\rho_T = \max(\rho_0, \ldots, \rho_{d-1})$. The type is given by the results of equations (26) and (27).

The Fourier transform is given by

$$U_F f(\{z_M\}) = \left[\mathcal{K}_0 E_{1/\rho_0}\left(\sigma_0^{1/\rho_0} \zeta_0\right)\right] \ldots \left[\mathcal{K}_{d-1} E_{1/\rho_{d-1}}\left(\sigma_{d-1}^{1/\rho_{d-1}} \zeta_{d-1}\right)\right], \tag{78}$$

where the ζ_M are related to the z_M through the Fourier transform of equation (50). The total growth of $U_F f(\{z_M\})$ is the same as the total growth of $f(\{z_M\})$.

10.1. Numerical results

We consider the case $d = 2$ with $\sigma_0 = 0.7$ and $\sigma_1 = 0.9$, i.e., the state

$$|s\rangle = |\rho_0, 0.7\rangle \otimes |\rho_1, 0.9\rangle. \tag{79}$$

We have studied numerically the properties of the states $|s\rangle$ and $U_F|s\rangle$.

Each of the two Hilbert spaces has been truncated at the number state $|N = 11\rangle$. To ensure that this is a good approximation, we continuously check that, for all states $|u\rangle$ involved in our calculations, their projections $|u\rangle_{\rm tr}$ into the truncated Hilbert space satisfy the

$${}_{\rm tr}\langle u|u\rangle_{\rm tr} > 0.9. \tag{80}$$

In other words, only a very small part of these states is outside the truncated Hilbert space. The fact that the average number of photons in these states (shown in figures 2 and 5) is much smaller than 11 also indicates that the approximation is good. We note here that as ρ increases a larger truncated Hilbert space is required for the same accuracy (and this is also consistent with the statement that $g^{(2)}$ is large at large ρ). For this reason, we present results for $\rho < 1.5$.

We first consider the state $|s\rangle$ and in figure 2 we plot the average numbers of photons $\langle N_0\rangle$ and $\langle N_1\rangle$ as functions of ρ_0 and ρ_1, correspondingly. In figure 3 we present the second-order correlations

$$g_0^{(2)} = \frac{\left\langle N_0^2\right\rangle - \langle N_0\rangle}{\langle N_0\rangle^2}, \qquad g_1^{(2)} = \frac{\left\langle N_1^2\right\rangle - \langle N_1\rangle}{\langle N_1\rangle^2} \tag{81}$$

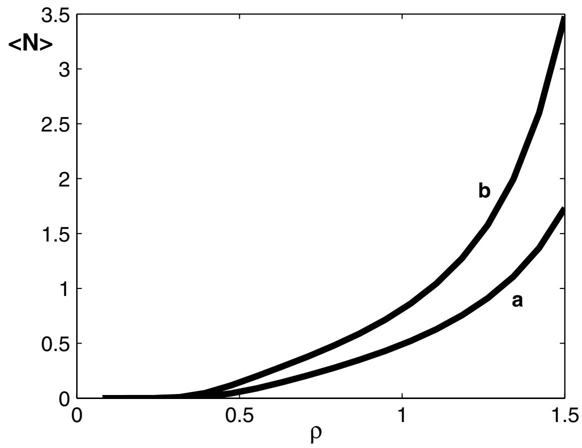

Figure 2. The average numbers of photons $\langle N_0 \rangle$ (curve a) and $\langle N_1 \rangle$ (curve b) for the state $|s\rangle$ of equation (79) as functions of the orders ρ_0 and ρ_1, correspondingly.

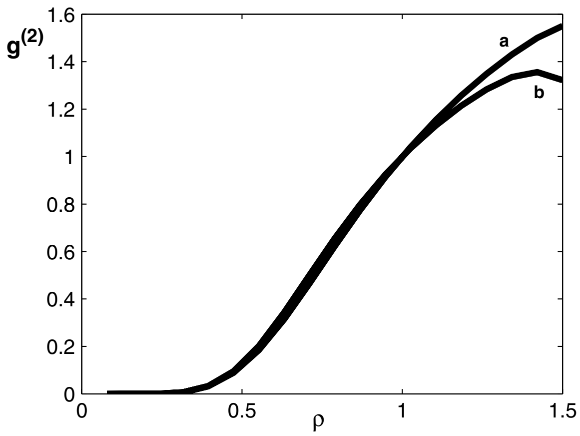

Figure 3. The second-order correlations $g_0^{(2)}$ (curve a) and $g_1^{(2)}$ (curve b) for the state $|s\rangle$ of equation (79) as functions of the orders ρ_0 and ρ_1, correspondingly.

as a function of ρ_0 and ρ_1, correspondingly. The state $|s\rangle$ is factorizable and therefore the results for the zero mode do not depend on ρ_1, and the results for the first mode do not depend on ρ_0.

We have also calculated the total number of photons $\langle N_T \rangle$ and the total second-order correlation $g_T^{(2)}$ of equation (28), as functions of ρ_0 and ρ_1, for the state $|s\rangle$. The intersection of the two-dimensional surface $\langle N_T \rangle$ with the plane at $\langle N_T \rangle = 0.62$ produced a curve on the plane $\rho_0 - \rho_1$. We considered the pairs (ρ_0, ρ_1) which belong to this curve and we calculated the corresponding $g_T^{(2)}$. We then plotted in figure 4 the $g_T^{(2)}$ as a function of the order of the total growth $\rho_T = \max(\rho_0, \rho_1)$. There is a multivaluedness in the results, which however does not affect our conclusions. We repeated the same calculation for $\langle N_T \rangle = 1.46$. The results support our earlier statement that if we compare states with the same total number of photons, the one with larger order of total growth, will also have larger total second-order correlation.

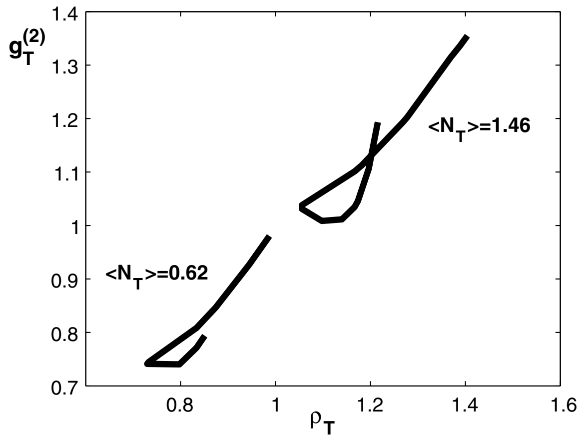

Figure 4. The total second-order correlations $g_T^{(2)}$ for the state $|s\rangle$ of equation (79) as a function of the order ρ_T for a total number of photons $\langle N_T\rangle = 0.62$ and $\langle N_T\rangle = 1.46$.

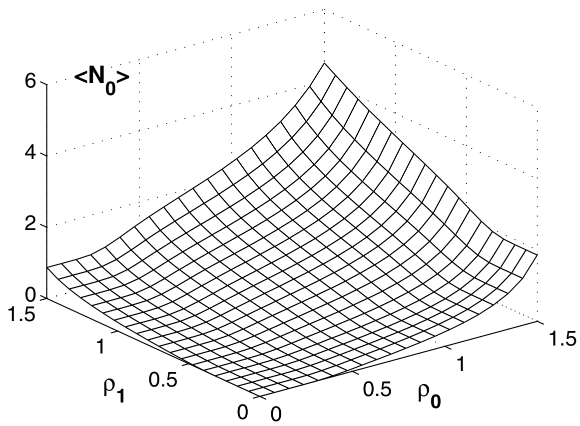

Figure 5. The average numbers of photons $\langle N_0\rangle$ for the state $U_F|s\rangle$ of equation (83) as a function of the orders ρ_0 and ρ_1.

In figures 5 and 6 we consider the transformed state $U_F|s\rangle$ and plot the average number of photons in the zero mode $\langle N_0\rangle$ and the corresponding second-order correlation $g_0^{(2)}$. The state $U_F|s\rangle$ is entangled and the results depend on both ρ_0 and ρ_1. In order to do the calculations, we use the formulae

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{82}$$

and find that

$$U_F|s\rangle = \sum g(M, N)|M, N\rangle \tag{83}$$

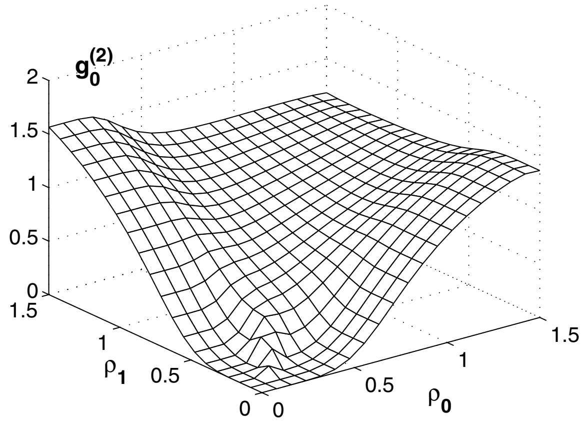

Figure 6. The second-order correlations $g_0^{(2)}$ for the state $U_F|s\rangle$ of equation (83) as a function of the orders ρ_0 and ρ_1.

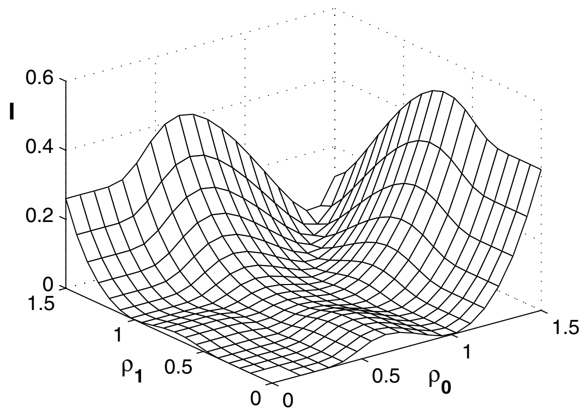

Figure 7. The entropic quantity I of equation (85) for the state $U_F|s\rangle$ as a function of the orders ρ_0 and ρ_1.

where

$$g(M,N) = \mathcal{K}_0\mathcal{K}_1(N!M!)^{1/2}\sum_{K,\Lambda}\frac{(-1)^{N-K}\left[2^{-1/2}\sigma_0^{1/\rho_0}\right]^{\Lambda}\left[2^{-1/2}\sigma_1^{1/\rho_1}\right]^{M+N-\Lambda}}{\Gamma\left(\frac{\Lambda}{\rho_0}+1\right)\Gamma\left(\frac{M+N-\Lambda}{\rho_1}+1\right)} \times \binom{\Lambda}{K}\binom{M+N-\Lambda}{N-K}. \qquad (84)$$

Here the integers M, N take all non-negative values.

As a measure of entanglement we present in figure 7 the quantity

$$I = S(\mathcal{R}_0) + S(\mathcal{R}_1) - S(\mathcal{R}_{01}), \qquad (85)$$

where $S(\mathcal{R})$ is the entropy of the density matrix $\mathcal{R}$,

$$S(\mathcal{R}) = -\mathrm{Tr}\,\mathcal{R}\ln\mathcal{R} \qquad (86)$$

and

$$\mathcal{R}_{01} = U_F|s\rangle\langle s|U_F^{\dagger}, \qquad \mathcal{R}_0 = \mathrm{Tr}_1\mathcal{R}_{01}, \qquad \mathcal{R}_1 = \mathrm{Tr}_0\mathcal{R}_{01}. \tag{87}$$

We have used natural logarithms and therefore the results are in nats.

11. Discussion

The Fourier transform of equation (48) is of great interest in quantum optics and for this reason it has been studied extensively in the literature. In this paper we have concentrated on theoretical aspects of this transform using the Bargmann representation which exploits the powerful formalism of analytic functions. Equation (50) gives the Fourier transform on an arbitrary state, in the Bargmann language.

We have discussed the growth of Bargmann functions and explained that it is related to the second-order correlation. We have shown that the Fourier transform and also the more general unitary transformations of equation (33) do not change the total growth of Bargmann functions. Two properties of the Fourier transform have been given in equations (53) and (54).

The Fourier transform of a coherent state is another coherent state. The Fourier transform of the factorizable squeezed state of equation (70) is the entangled squeezed state of equation (71).

The Mittag–Leffler states of equation (76) demonstrate that we can have states of any given growth, provided that $\rho_T < 2$ (in which case σ_T can have any value), or $\rho_T = 2$ and $\sigma_T < 1/2$. Physical quantities for these states have been presented in figures 2–4, and for the Fourier transform of these states in figures 5–7.

In summary, the Fourier transform of equation (48) has been studied extensively in the literature. This work is complemented in the present paper by studying this transform from the point of view of the theory of analytic functions of many complex variables.

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